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Interaction of noncommutative plane waves in $(2 + 1)$ dimensions

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Abstract

In this paper the interactions of extended waves in a noncommutative modified $(2 + 1)$ -dimensional $U(2)$ sigma model are studied. Using the dressing method, we construct an explicit two-wave solution of the noncommutative field equation. The scattering of these waves and large-time factorization are discussed.

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1. Introduction

Possible noncommutativity of spacetime coordinates is an old idea [1] which was recently revived in the active study of noncommutative field theories (see, e.g., reviews [2–5] and references therein). Being nonlocal, these theories have many interesting properties.

It has been found that noncommutative field theories arise naturally in certain limits of string theories involving D -branes and a nontrivial B -field background [6]. Recently, this fact led to a high level of interest in field theories on noncommutative spaces and their nonperturbative solutions (like solitons), which admit a D -brane interpretation in the context of string theory [2, 3, 7, 8]. A lot of progress has been made in the analysis of such solitonic solutions of various noncommutative field theories (see, e.g., [2–4, 7–18] and references therein).

In [19], it was shown that the open $N = 2$ fermionic string is identical at tree level to self-dual Yang–Mills theory in $(2 + 2)$ dimensions. Turning on a constant B -field in this string theory yields *noncommutative* self-dual Yang–Mills (ncSDYM) [20]. If the B -field is restricted to the world volume of n coincident $D2$ -branes the effective field theory turns out to be the noncommutative generalization [21] of an integrable modified $U(n)$ sigma model in $(2 + 1)$ dimensions introduced by Ward [22]. Its soliton solutions have been studied in [16].

In the present paper we examine a different type of solutions of the noncommutative modified sigma model mentioned above, namely plane waves, which were studied in the

commutative set-up by Leese [23]¹. Our work is based on [16], where the gauge group is restricted to the case of $U(2)$. The organization is the following. After presenting the necessary material and formulae for the commutative sigma model we briefly review its noncommutative extension and the general construction of solutions. We then derive an explicit two-wave solution and investigate its scattering properties and large-time factorization.

2. Noncommutative modified $U(2)$ sigma model

The investigations are based on a modified $SU(2)$ sigma model in $(2 + 1)$ dimensions introduced by Ward [22]. This model describes the dynamics of an $SU(2)$ -valued field $J(t, x, y)$ depending on the real coordinates t, x, y of $\mathbb{R}^{1,2}$ with Minkowskian signature $(- + +)$. Its field equation reads

$$\eta^{\mu\nu} \partial_\mu (J^{-1} \partial_\nu J) + v_\alpha \epsilon^{\alpha\mu\nu} \partial_\mu (J^{-1} \partial_\nu J) = 0 \quad (2.1)$$

with a constant unit vector $(v_\alpha) = (0, 1, 0)$. It is obvious that the second term in (2.1) breaks Lorentz invariance, but makes this equation integrable [22]. The model has a positive definite energy functional, which is given by

$$E = \int dx dy \mathcal{E} = \frac{1}{2} \int dx dy \operatorname{tr} (\partial_t J^{-1} \partial_t J + \partial_x J^{-1} \partial_x J + \partial_y J^{-1} \partial_y J). \quad (2.2)$$

The field equation (2.1) actually arises from the Bogomol'nyi equations for the $(2 + 1)$ -dimensional Yang–Mills–Higgs system [26]

$$\frac{1}{2} \epsilon_{\alpha\mu\nu} \mathcal{F}^{\mu\nu} = \partial_\alpha \Phi + [\mathcal{A}_\alpha, \Phi]. \quad (2.3)$$

After choosing the gauge $\mathcal{A}_x = -\Phi$ and $\mathcal{A}_t = \mathcal{A}_y$ and solving two of the three equations above by $\mathcal{A}_x = \frac{1}{2} J^{-1} \partial_x J$ and $\mathcal{A}_t = \frac{1}{2} J^{-1} (\partial_t J + \partial_y J)$ one obtains

$$\partial_x (J^{-1} \partial_x J) + \partial_y (J^{-1} \partial_y J) - \partial_t (J^{-1} \partial_t J) + \partial_y (J^{-1} \partial_t J) - \partial_t (J^{-1} \partial_y J) = 0 \quad (2.4)$$

which is precisely equation (2.1). It admits both soliton [22] and extended wave solutions [23].

The noncommutative generalization of equation (2.1) is easily achieved by deforming the ordinary product of functions into the noncommutative star product, which is given by

$$(f \star g)(x) = f(x) \exp \left[\frac{i}{2} \overleftarrow{\partial}_\mu \theta^{\mu\nu} \overrightarrow{\partial}_\nu \right] g(x). \quad (2.5)$$

Since the time coordinate remains commutative, the only nonvanishing components of the tensor $\theta^{\mu\nu}$ are

$$\theta^{xy} = -\theta^{yx} =: \theta > 0. \quad (2.6)$$

After introducing the following combinations of the coordinates t and y

$$u := \frac{1}{2}(t + y), \quad v := \frac{1}{2}(t - y) \quad \Rightarrow \quad \partial_u = \partial_t + \partial_y, \quad \partial_v = \partial_t - \partial_y \quad (2.7)$$

the noncommutative field equation² reads [21]

$$\partial_x (J^{-1} \star \partial_x J) - \partial_v (J^{-1} \star \partial_u J) = 0 \quad (2.8)$$

and the energy density is

$$\mathcal{E} = \frac{1}{2} \operatorname{tr} [\partial_t J^\dagger \star \partial_t J + \partial_x J^\dagger \star \partial_x J + \partial_y J^\dagger \star \partial_y J]. \quad (2.9)$$

¹ For plane wave solutions of the self-dual Yang–Mills equation in $(3 + 1)$ dimensions see, e.g., [24, 25].

² It should be noted here that in the noncommutative set-up $J \in U(2)$, since $SU(2)$ as gauge group is not possible. See, for example, [27, 28].

The nonlocality of the star product involves difficulties in explicit calculations. It is therefore more convenient to switch to the operator formalism. Here, the star product is traded for operator-valued spatial coordinates \hat{x} , \hat{y} , which are subject to the following commutation relations

$$[t, \hat{x}] = [t, \hat{y}] = 0 \quad [\hat{x}, \hat{y}] = i\theta. \quad (2.10)$$

With the complex coordinates, $\hat{z} = \hat{x} + i\hat{y}$, $\hat{\bar{z}} = \hat{x} - i\hat{y}$, relation (2.10) becomes $[\hat{z}, \hat{\bar{z}}] = 2\theta$. Now one can introduce creation and annihilation operators

$$a^\dagger = \frac{1}{\sqrt{2\theta}} \hat{\bar{z}} \quad a = \frac{1}{\sqrt{2\theta}} \hat{z} \quad \Rightarrow \quad [a, a^\dagger] = 1 \quad (2.11)$$

which act in a harmonic oscillator Hilbert space \mathcal{H} with orthonormal basis $\{|n\rangle, n = 0, 1, 2, \dots\}$ in standard fashion

$$a|n\rangle = \sqrt{n}|n-1\rangle \quad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle. \quad (2.12)$$

Every c-number function $f(t, z, \bar{z})$ can be associated with an operator-valued function $\hat{f} \equiv F(t, a, a^\dagger)$ through the Moyal–Weyl map

$$\begin{aligned} f(t, z, \bar{z}) \rightarrow F(t, a, a^\dagger) &= \frac{1}{(2\pi)^2} \int d^2k \tilde{f}(k_x, k_y) \exp(-i(k_x \hat{x} + k_y \hat{y})) \\ &= - \int \frac{dp d\bar{p}}{(2\pi^2)} dz d\bar{z} f(t, z, \bar{z}) \exp(-i[\bar{p}(\sqrt{2\theta}a - z) + p(\sqrt{2\theta}a^\dagger - \bar{z})]) \end{aligned} \quad (2.13)$$

with $p = \frac{1}{2}(k_x + ik_y)$. The inverse transformation reads

$$f(t, z, \bar{z}) = F_\star \left(t, \frac{z}{\sqrt{2\theta}}, \frac{\bar{z}}{\sqrt{2\theta}} \right) \quad (2.14)$$

where F_\star is obtained by replacing ordinary products with star products. Under the Moyal–Weyl map one has [8, 29]

$$\partial_z \rightarrow \hat{\partial}_z = -\frac{1}{\sqrt{2\theta}} [a^\dagger,] \quad \partial_{\bar{z}} \rightarrow \hat{\partial}_{\bar{z}} = \frac{1}{\sqrt{2\theta}} [a,] \quad (2.15)$$

$$f \star g \rightarrow \hat{f} \hat{g} \quad \int dx dy f = 2\pi\theta \text{Tr}_{\mathcal{H}} \hat{f} = 2\pi\theta \sum_{n \geq 0} \langle n | \hat{f} | n \rangle. \quad (2.16)$$

Hence, in terms of creation and annihilation operators the energy density (2.9) becomes

$$\hat{\mathcal{E}} = \frac{1}{2} \text{tr} [\partial_t \hat{J}^\dagger \partial_t \hat{J}] + \frac{1}{2\theta} \text{tr} [[a, \hat{J}^\dagger][a, \hat{J}^\dagger]^\dagger + [a^\dagger, \hat{J}^\dagger][a^\dagger, \hat{J}^\dagger]^\dagger] \quad (2.17)$$

where $\hat{\partial}_x = \hat{\partial}_z + \hat{\partial}_{\bar{z}}$ and $\hat{\partial}_y = i(\hat{\partial}_z - \hat{\partial}_{\bar{z}})$ have been used, and $\hat{J} \equiv J(t, a, a^\dagger)$. For simplicity, hats over operators are omitted from now on.

3. Construction of solutions

The main observation for constructing solutions of the equation of motion (2.8) is that this equation can be formulated as the compatibility condition of a system of linear differential equations. For the commutative model the ‘dressing method’ [30, 31] can be used to construct solitons and plane waves as shown in [22, 23]. This can easily be extended to the noncommutative situation, which will be briefly presented next. For more details see [16].

Let us consider the following linear system

$$(\zeta \partial_x - \partial_u) \psi = A \psi \quad (\zeta \partial_v - \partial_x) \psi = B \psi \quad (3.1)$$

where $\psi = \psi(x, u, v, \zeta)$ is a 2×2 matrix whose elements are operators acting in \mathcal{H} . The matrices A and B are of the same type but do not depend on the parameter $\zeta \in \mathbb{C}$. The matrix ψ has to satisfy the reality condition [22]

$$\psi(x, u, v, \zeta) [\psi(x, u, v, \bar{\zeta})]^\dagger = 1. \quad (3.2)$$

The integrability conditions for system (3.1) are

$$\partial_x B - \partial_v A = 0 \quad \partial_x A - \partial_u B - [A, B] = 0. \quad (3.3)$$

The second equation can be solved by parametrizing the matrices A and B in the following way:

$$A = J^{-1} \partial_u J \quad B = J^{-1} \partial_x J \quad (3.4)$$

where J is some unitary 2×2 matrix. Then the first equation in (3.3) becomes

$$\partial_x (J^{-1} \partial_x J) - \partial_v (J^{-1} \partial_u J) = 0 \quad (3.5)$$

which is the operator version of the field equation (2.8). Thus, any solution of the system (3.1) yields a solution of (3.5) since

$$\psi(x, u, v, \zeta = 0) = J^{-1}(x, u, v). \quad (3.6)$$

Let us now assume that ψ is of the form [30, 31, 22, 16]

$$\psi = 1 + \sum_{p=1}^m \frac{R_p}{\zeta - \mu_p} \quad R_p = \sum_{l=1}^m T_l \Gamma^{lp} T_p^\dagger \quad (3.7)$$

where μ_p are complex constants, the $T_k(t, x, y)$ are some 2×1 matrices and the Γ^{kl} are some operator-valued functions. By multiplying (3.1) with ψ^{-1} on the right and using (3.2), the linear system can be rewritten as

$$-\psi(\zeta)(\zeta \partial_x - \partial_u)\psi(\bar{\zeta})^\dagger = A \quad -\psi(\zeta)(\zeta \partial_v - \partial_x)\psi(\bar{\zeta})^\dagger = B. \quad (3.8)$$

The singularities at $\zeta = \mu$ and $\zeta = \bar{\mu}$ have to be removable, since the right-hand side of these equations does not depend on ζ . By putting to zero the corresponding residues one finds two differential equations

$$\left(1 - \sum_{p=1}^m \frac{R_p}{\mu_p - \bar{\mu}_k}\right) (\bar{\mu}_k \partial_x - \partial_u) R_k^\dagger = 0 \quad \left(1 - \sum_{p=1}^m \frac{R_p}{\mu_p - \bar{\mu}_k}\right) (\bar{\mu}_k \partial_v - \partial_x) R_k^\dagger = 0. \quad (3.9)$$

In the same manner, one obtains from (3.2)

$$\left(1 - \sum_{p=1}^m \frac{R_p}{\mu_p - \bar{\mu}_k}\right) T_k = 0 \quad (3.10)$$

which yields the inverse of the functions Γ^{kl} in terms of the matrices T_k [16],

$$\tilde{\Gamma}^{kl} = \frac{T_k^\dagger T_l}{\mu_k - \bar{\mu}_l} \quad \text{and} \quad \sum_{p=1}^m \Gamma^{lp} \tilde{\Gamma}^{pk} = \delta_k^l. \quad (3.11)$$

It is useful to introduce the following new coordinates

$$\begin{aligned} w_k &:= v_k \left(x + \bar{\mu}_k u + \frac{1}{\bar{\mu}_k} v \right) = v_k \left[x + \frac{1}{2} \left(\bar{\mu}_k - \frac{1}{\bar{\mu}_k} \right) y + \frac{1}{2} \left(\bar{\mu}_k + \frac{1}{\bar{\mu}_k} \right) t \right] \\ \bar{w}_k &:= \bar{v}_k \left(x + \mu_k u + \frac{1}{\mu_k} v \right) = \bar{v}_k \left[x + \frac{1}{2} \left(\mu_k - \frac{1}{\mu_k} \right) y + \frac{1}{2} \left(\mu_k + \frac{1}{\mu_k} \right) t \right] \end{aligned} \quad (3.12)$$

with the v_k being functions of the μ_k ,

$$v_k = \left[\frac{4i}{\mu_k - \bar{\mu}_k - \mu_k^{-1} + \bar{\mu}_k^{-1}} \frac{\mu_k - \mu_k^{-1} - 2i}{\bar{\mu}_k - \bar{\mu}_k^{-1} + 2i} \right]^{1/2}. \quad (3.13)$$

Since

$$[w_k, \bar{w}_k] = 2\theta \quad (3.14)$$

one can define the ‘co-moving’ creation and annihilation operators

$$c_k = \frac{1}{\sqrt{2\theta}} w_k, \quad c_k^\dagger = \frac{1}{\sqrt{2\theta}} \bar{w}_k \quad \Rightarrow \quad [c_k, c_k^\dagger] = 1 \quad (3.15)$$

with oscillator basis $\{|n\rangle_k\}$. Derivatives with respect to these coordinates are represented in the same way as in (2.15),

$$\sqrt{2\theta} \partial_{w_k} = -[c_k^\dagger,] \quad \sqrt{2\theta} \partial_{\bar{w}_k} = [c_k,]. \quad (3.16)$$

One can relate the co-moving oscillators to the static one. Expressing the w_k through z and \bar{z} , one obtains

$$c_k = (\cosh \tau_k) a - (e^{i\vartheta_k} \sinh \tau_k) a^\dagger - \beta_k t \quad (3.17)$$

where $v_k = \cosh \tau_k - e^{i\vartheta_k} \sinh \tau_k$, $e^{i\vartheta_k} \tanh \tau_k = \frac{\bar{\mu}_k - \bar{\mu}_k^{-1} - 2i}{\bar{\mu}_k - \bar{\mu}_k^{-1} + 2i}$ and

$$\beta_k := -\frac{1}{2}(2\theta)^{-1/2} v_k (\bar{\mu}_k + \bar{\mu}_k^{-1}). \quad (3.18)$$

Equations (3.9) can now be combined to a single one,

$$\left(1 - \sum_{p=1}^m \frac{R_p}{\mu_p - \bar{\mu}_k} \right) c_k T_k = 0. \quad (3.19)$$

With (3.10) it is quite obvious that a sufficient condition for a solution is

$$c_k T_k = T_k Z_k \quad (3.20)$$

with some operator Z_k . If Z_k is taken to be c_k then (3.20) becomes

$$[c_k, T_k] = 0 \quad (3.21)$$

which means that any matrix T_k with elements being arbitrary functions of c_k yields a solution ψ of the linear system (3.1), and by (3.6) a solution J of the equation of motion (3.5).

4. A two-wave configuration

As was shown in the previous section, the solution for $J^{-1} = J^\dagger$ can be expressed through R_k

$$J^\dagger = 1 - \sum_{k=1}^m \frac{R_k}{\mu_k}. \quad (4.1)$$

In order to obtain an extended wave solution comparable to the commutative solution of Leese, the elements of the T_k are taken to be exponentials of $b_k w_k = b_k \sqrt{2\theta} c_k$,³ with $b_k \in \mathbb{R}$. To be specific

$$T_k = \begin{pmatrix} 1 \\ e^{b_k \omega_k} \end{pmatrix}. \quad (4.2)$$

³ Compare with [23].

From now on let us take $m = 2$ in formula (4.1),

$$J^\dagger = 1 - \frac{1}{\mu_1} T_1 \Gamma^{11} T_1^\dagger - \frac{1}{\mu_2} T_1 \Gamma^{12} T_2^\dagger - \frac{1}{\mu_1} T_2 \Gamma^{21} T_1^\dagger - \frac{1}{\mu_2} T_2 \Gamma^{22} T_2^\dagger. \quad (4.3)$$

Now the functions Γ^{kl} can be expressed in terms of $\tilde{\Gamma}_{lp}$ via (3.11),

$$\begin{aligned} \Gamma^{11} &= [\tilde{\Gamma}_{11} - \tilde{\Gamma}_{12} \tilde{\Gamma}_{22}^{-1} \tilde{\Gamma}_{21}]^{-1} \\ \Gamma^{12} &= -[\tilde{\Gamma}_{11} - \tilde{\Gamma}_{12} \tilde{\Gamma}_{22}^{-1} \tilde{\Gamma}_{21}]^{-1} \tilde{\Gamma}_{12} \tilde{\Gamma}_{22}^{-1} \\ \Gamma^{21} &= -[\tilde{\Gamma}_{22} - \tilde{\Gamma}_{21} \tilde{\Gamma}_{11}^{-1} \tilde{\Gamma}_{12}]^{-1} \tilde{\Gamma}_{21} \tilde{\Gamma}_{11}^{-1} \\ \Gamma^{22} &= [\tilde{\Gamma}_{22} - \tilde{\Gamma}_{21} \tilde{\Gamma}_{11}^{-1} \tilde{\Gamma}_{12}]^{-1} \end{aligned} \quad (4.4)$$

which yields J^\dagger in explicit T_k -dependence,

$$\begin{aligned} J^\dagger &= 1 - T_1 [T_1^\dagger (1 - \sigma P_2) T_1]^{-1} T_1^\dagger \left(\frac{\mu_{11}}{\mu_1} - \frac{\mu_{21}}{\mu_2} \sigma P_2 \right) \\ &\quad - T_2 [T_2^\dagger (1 - \sigma P_1) T_2]^{-1} T_2^\dagger \left(\frac{\mu_{22}}{\mu_2} - \frac{\mu_{12}}{\mu_1} \sigma P_1 \right) \end{aligned} \quad (4.5)$$

where the following complex parameters have been introduced:

$$\mu_{kl} = \mu_k - \bar{\mu}_l \quad \sigma = \frac{\mu_{11} \mu_{22}}{\mu_{12} \mu_{21}}. \quad (4.6)$$

The P_k are Hermitian projectors parametrized through the matrices T_k ,

$$P_k = T_k (T_k^\dagger T_k)^{-1} T_k^\dagger \quad P_k^\dagger = P_k \quad P_k^2 = P_k. \quad (4.7)$$

For simplicity, we choose the first wave to be static, $\mu_1 = -i$, which means $w_1 = z$, and rename $\mu_2 := \mu$, $w_2 := w$. In addition, μ is restricted to be pure imaginary, $\mu = ip$, with real $p > 1$. This ensures the existence of large-time limits of $\exp(b_2 w)$ and $\exp(b_2 \bar{w})$, which will be important in further considerations. The strategy is to evaluate the energy density \mathcal{E} for large positive and large negative times and investigate the interaction.

For $\mu = ip$ the constant β in (3.17) will be real,

$$\beta = -\frac{1}{2\sqrt{\theta}} \frac{p - \frac{1}{p}}{\sqrt{p + \frac{1}{p}}} < 0 \quad (4.8)$$

so that large-time limits can be easily evaluated. For t being large means⁴ $b_2 w = b_2 \sqrt{2\theta} c \rightarrow +b_2 \sqrt{2\theta} |\beta| t$, and finally $e^{b_2 w} \rightarrow +\infty$, $e^{b_2 \bar{w}} \rightarrow +\infty$. In this limit, J^\dagger becomes⁵ ($b := b_1$)

$$\begin{aligned} J_{+\infty}^\dagger &:= \lim_{t \rightarrow +\infty} J^\dagger \\ &= 1 - 2 \begin{pmatrix} [1 + (1 - \sigma) e^{b\bar{z}} e^{bz}]^{-1} & -\frac{1+p}{1-p} [1 + (1 - \sigma) e^{b\bar{z}} e^{bz}]^{-1} e^{b\bar{z}} \\ \frac{1+p}{1-p} [1 + (1 - \sigma) e^{bz} e^{b\bar{z}}]^{-1} e^{bz} & [1 + (1 - \sigma) e^{bz} e^{b\bar{z}}]^{-1} \end{pmatrix}. \end{aligned} \quad (4.9)$$

For large negative times, $e^{b_2 w} \rightarrow 0$, $e^{b_2 \bar{w}} \rightarrow 0$, and we obtain ($b := b_1$)

$$\begin{aligned} J_{-\infty}^\dagger &:= \lim_{t \rightarrow -\infty} J^\dagger \\ &= -1 - 2 \begin{pmatrix} -(1 - \sigma) [1 - \sigma + e^{b\bar{z}} e^{bz}]^{-1} & \frac{1+p}{1-p} [1 - \sigma + e^{b\bar{z}} e^{bz}]^{-1} e^{b\bar{z}} \\ -\frac{1+p}{1-p} [1 - \sigma + e^{bz} e^{b\bar{z}}]^{-1} e^{bz} & -(1 - \sigma) [1 - \sigma + e^{bz} e^{b\bar{z}}]^{-1} \end{pmatrix} \end{aligned} \quad (4.10)$$

⁴ b_2 is restricted to be greater than zero, in order to achieve a well-defined limit.

⁵ We refrain from writing down the full expression for J^\dagger .

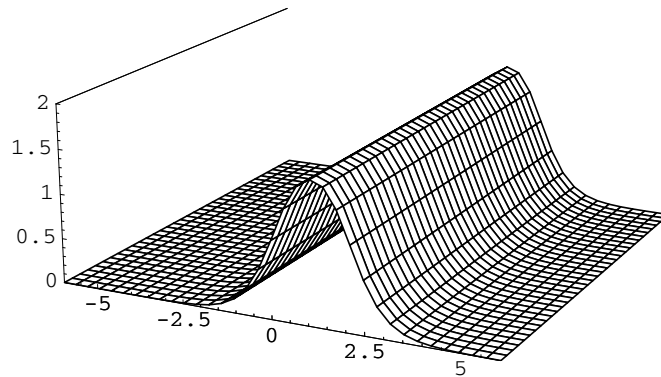


Figure 1. $\mathcal{E}_{-\infty, \star}$ with $b = 1$, $p = 2$, $\theta = 1$.

where the first sign results from simplifying matrix elements. Now the energy density can be readily calculated.

For large times, $J \rightarrow J_0 + \mathcal{O}(t^{-1})$, with J_0 being independent of t , and therefore in this limit the first term in equation (2.17) vanishes. Hence, the asymptotics of the energy density operator are⁶

$$\hat{\mathcal{E}}_{\pm\infty} := \lim_{t \rightarrow \pm\infty} \hat{\mathcal{E}} = \frac{1}{2\theta} \text{tr}[[a, \hat{J}_{\pm\infty}^\dagger][a, \hat{J}_{\pm\infty}^\dagger]^\dagger + [a^\dagger, \hat{J}_{\pm\infty}^\dagger][a^\dagger, \hat{J}_{\pm\infty}^\dagger]^\dagger] \quad (4.11)$$

with $\hat{J}_{\pm\infty}^\dagger$ given by (4.9) and (4.10). By noting that $e^{b\hat{z}} e^{b\hat{z}} \rightarrow e^{2bx - b^2\theta}$ and that the star product of two functions depending only on x is identical to usual multiplication, one sees that the final result in star product formulation is

$$\begin{aligned} \mathcal{E}_{-\infty, \star} &= 4b^2 \left\{ \text{sech}^2 \left[bx - \frac{b^2\theta}{2} + \gamma \right] + \text{sech}^2 \left[bx + \frac{b^2\theta}{2} + \gamma \right] \right\} \\ \mathcal{E}_{+\infty, \star} &= 4b^2 \left\{ \text{sech}^2 \left[bx - \frac{b^2\theta}{2} - \gamma \right] + \text{sech}^2 \left[bx + \frac{b^2\theta}{2} - \gamma \right] \right\} \end{aligned} \quad (4.12)$$

with γ being a real constant depending only on p ,

$$\gamma = -\frac{1}{2} \ln \left(\frac{1+p}{1-p} \right)^2. \quad (4.13)$$

These two noncommutative plane waves interact in a simple way (see figures 1–4). The phase of the static wave depends on the noncommutativity parameter θ . This wave experiences a phase shift of 2γ , which is independent of θ and precisely matches the result obtained from the commutative model by Leese⁷. The phase shift is clearly visible in the figures. Moreover, it is easy to see that in the commutative limit, $\theta \rightarrow 0$, formulae (4.12) recover the energy density as obtained by Leese⁸.

⁶ To avoid any confusion, operators are marked again with hats.

⁷ See [23] and formula (4.5) for $p_1 = -1$ and $p_2 = p$.

⁸ See [23] and formula (4.4).

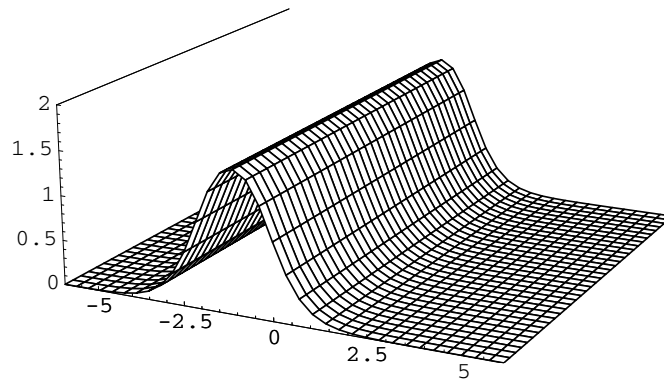


Figure 2. $\mathcal{E}_{+\infty, \star}$ with $b = 1$, $p = 2$, $\theta = 1$.

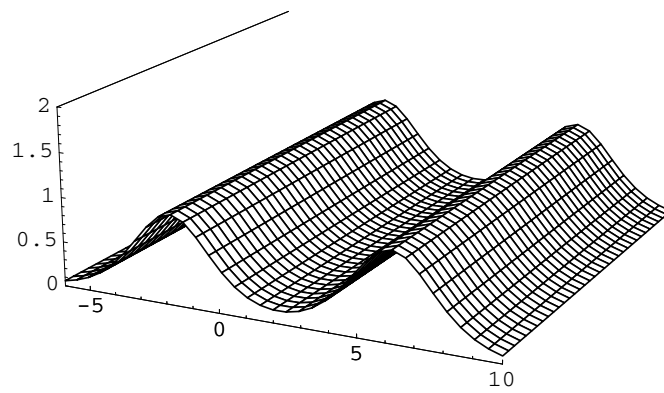


Figure 3. $\mathcal{E}_{-\infty, \star}$ with $b = 1$, $p = 2$, $\theta = 4$.

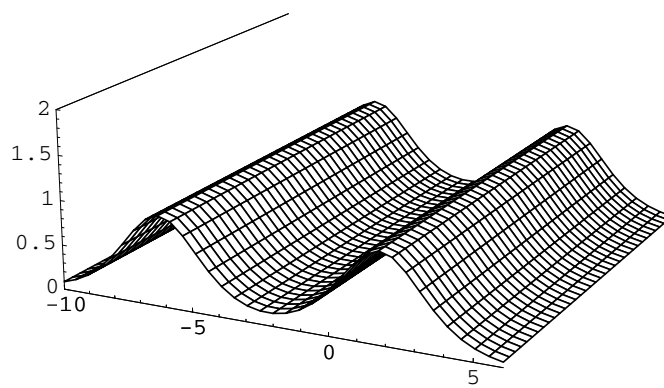


Figure 4. $\mathcal{E}_{+\infty, \star}$ with $b = 1$, $p = 2$, $\theta = 4$.

5. Large-time factorization

Besides the additive ansatz used in (4.3), one can also use a multiplicative ansatz. Let us take the function $\hat{\psi}(t, \hat{x}, \hat{y}, \zeta)$ to be of the form

$$\hat{\psi}_2 = \left(1 + \frac{-2i}{\zeta + i} \tilde{P}_1\right) \left(1 + \frac{\mu - \bar{\mu}}{\zeta - \mu} \tilde{P}_2\right) \quad (5.1)$$

where the operators \tilde{P}_1 and \tilde{P}_2 are yet to be determined. Because the reality condition (3.2) has to hold, \tilde{P}_1 and \tilde{P}_2 are identified as Hermitian projectors, parametrized as usual, $\tilde{P}_k = \tilde{T}_k (\tilde{T}_k^\dagger \tilde{T}_k)^{-1} \tilde{T}_k^\dagger$, $k = 1, 2$, with some 2×1 matrices \tilde{T}_k . The removability of the singularities at $\zeta = \bar{\mu}$ and $\zeta = \mu$ in (3.8) is guaranteed if \tilde{P}_2 satisfies the equation

$$(1 - \tilde{P}_2) c \tilde{P}_2 = 0 \quad \Rightarrow \quad c \tilde{T}_2 = \tilde{T}_2 Z_2 \quad (5.2)$$

which means, for $Z_2 = c$, that the elements of \tilde{T}_2 are functions of c . For the residues at $\zeta = \pm i$, one obtains the equation

$$\frac{1}{\sqrt{2\theta}} (1 - \tilde{P}_1) \left(1 + \frac{\mu - \bar{\mu}}{i - \mu} \tilde{P}_2\right) a \left(1 + \frac{\bar{\mu} - \mu}{i - \bar{\mu}} \tilde{P}_2\right) \tilde{P}_1 = 0. \quad (5.3)$$

This equation looks more complicated than (5.2), but can be investigated in large positive and large negative time limits, yielding an equation similar to (5.2). Let \tilde{T}_2 , solving equation (5.2), be of the form

$$\tilde{T}_2 = \begin{pmatrix} \gamma_1 \\ \gamma_2 e^{\gamma_3 c} \end{pmatrix} \quad (5.4)$$

with constant parameters γ_k , $k = 1, 2, 3$, and $\gamma_3 > 0$. Then, for $t \rightarrow \pm\infty$, one gets

$$\lim_{t \rightarrow -\infty} \tilde{P}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} =: \Pi_{-\infty} \quad \lim_{t \rightarrow +\infty} \tilde{P}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} =: \Pi_{+\infty}. \quad (5.5)$$

So, in these limits \tilde{P}_2 is a constant projector. Therefore, the operator a in equation (5.3) can be moved to the left of \tilde{P}_1 and (5.3) simplifies to $(\tilde{P}_{1,\pm\infty} := \lim_{t \rightarrow \pm\infty} \tilde{P}_1)$

$$(1 - \tilde{P}_{1,\pm\infty}) a \tilde{P}_{1,\pm\infty} = 0 \quad \Rightarrow \quad a \tilde{T}_{1,\pm\infty} = \tilde{T}_{1,\pm\infty} Z_1. \quad (5.6)$$

With $Z_1 = a$, the elements of $\tilde{T}_{1,\pm\infty}$ are arbitrary functions of a , and in this case $\tilde{T}_{1,\pm\infty}$ can be chosen to be

$$\tilde{T}_{1,-\infty} = \begin{pmatrix} \lambda_1 \\ \lambda_2 e^{\lambda_3 a} \end{pmatrix} \quad \tilde{T}_{1,+\infty} = \begin{pmatrix} \lambda_4 \\ \lambda_5 e^{\lambda_6 a} \end{pmatrix} \quad (5.7)$$

where the λ_k are yet to be determined. Writing

$$\hat{J}_{\pm\infty}^\dagger = \lim_{t \rightarrow \pm\infty} \hat{\psi}_2(\zeta = 0) = (1 - 2\tilde{P}_{1,\pm\infty})(1 - 2\Pi_{\pm\infty}) \quad (5.8)$$

for $\mu = ip$ and comparing (5.7) with formulae (4.9) and (4.10), one finds

$$\tilde{T}_{1,-\infty} = \begin{pmatrix} \lambda \\ e^{b\hat{z}} \end{pmatrix} = [1 + (\lambda - 1)\Pi_{-\infty}]T_1 \quad \tilde{T}_{1,+\infty} = \begin{pmatrix} 1 \\ \lambda e^{b\hat{z}} \end{pmatrix} = [1 + (\lambda - 1)\Pi_{+\infty}]T_1 \quad (5.9)$$

where $\lambda := \frac{1+p}{1-p}$ and T_1 is given in (4.2). So, \hat{J}^\dagger factorizes in the large time limit in a term $(1 - 2\tilde{P}_{1,\pm\infty})$ constructed from a modified time-dependent one-wave solution ($T_1 \rightarrow \tilde{T}_{1,\pm\infty}$)

and a constant unitary matrix $(1 - 2\Pi_{\pm\infty}) =: U_{\pm\infty}^\dagger$. The energy density $\hat{\mathcal{E}}_{\pm\infty}$ from equation (2.17) now reads

$$\begin{aligned}\hat{\mathcal{E}}_{\pm\infty} &= \frac{1}{2\theta} \text{tr} [[a, (1 - 2\tilde{P}_{1,\pm\infty})U_{\pm\infty}^\dagger][a, (1 - 2\tilde{P}_{1,\pm\infty})U_{\pm\infty}^\dagger]^\dagger \\ &\quad + [a^\dagger, (1 - 2\tilde{P}_{1,\pm\infty})U_{\pm\infty}^\dagger][a^\dagger, (1 - 2\tilde{P}_{1,\pm\infty})U_{\pm\infty}^\dagger]^\dagger] \\ &= \frac{2}{\theta} \text{tr} [[a, \tilde{P}_{1,\pm\infty}][a, \tilde{P}_{1,\pm\infty}]^\dagger + [a^\dagger, \tilde{P}_{1,\pm\infty}][a^\dagger, \tilde{P}_{1,\pm\infty}]^\dagger]\end{aligned}\quad (5.10)$$

where the projectors $\tilde{P}_{1,\pm\infty}$ are given by

$$\begin{aligned}\tilde{P}_{1,-\infty} &= \begin{pmatrix} (1 - \sigma)[(1 - \sigma) + e^{b\hat{z}} e^{b\hat{z}}]^{-1} & \frac{1+p}{1-p} [(1 - \sigma) + e^{b\hat{z}} e^{b\hat{z}}]^{-1} e^{b\hat{z}} \\ \frac{1+p}{1-p} [(1 - \sigma) + e^{b\hat{z}} e^{b\hat{z}}]^{-1} e^{b\hat{z}} & 1 - [(1 - \sigma) + e^{b\hat{z}} e^{b\hat{z}}]^{-1} \end{pmatrix} \\ \tilde{P}_{1,+\infty} &= \begin{pmatrix} [1 + (1 - \sigma) e^{b\hat{z}} e^{b\hat{z}}]^{-1} & \frac{1+p}{1-p} [1 + (1 - \sigma) e^{b\hat{z}} e^{b\hat{z}}]^{-1} e^{b\hat{z}} \\ \frac{1+p}{1-p} [1 + (1 - \sigma) e^{b\hat{z}} e^{b\hat{z}}]^{-1} e^{b\hat{z}} & (1 - \sigma) \{1 - [1 + (1 - \sigma) e^{b\hat{z}} e^{b\hat{z}}]^{-1}\} \end{pmatrix}.\end{aligned}\quad (5.11)$$

By construction the operators $\hat{\mathcal{E}}_{\pm\infty}$ from (5.10) are identical to those in equation (2.17) with $\hat{J}_{\pm\infty}^\dagger$ given by (4.9) and (4.10). So, both additive and multiplicative ansätze result in the same energy densities $\mathcal{E}_{\pm\infty,*}$ given by formulae (4.12).

6. Conclusion

In this paper, we have constructed an explicit time-dependent two-wave solution of a noncommutative $(2 + 1)$ -dimensional modified $U(2)$ sigma model. We have shown that this configuration depends on the noncommutativity parameter θ , but the phase shift produced by the interaction is independent of θ , coinciding with the commutative case. We have also proven that this solution factorizes in the large-time limit.

As was previously remarked, the considered noncommutative modified sigma model can be obtained from ncSDYM in $(2 + 2)$ dimensions [20]. For the commutative case it is well known that most integrable equations in three or less dimensions derive from SDYM by suitable dimensional reductions [32–36]. It will therefore be interesting to consider other reductions of ncSDYM (see, e.g., [37–40]) and to construct solutions of those noncommutative integrable equations with the help of the dressing approach.

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